BERNARD BOLZANO.*

(1781-1848.)

IN BOLZANO we find the virtues of human sympathy and insight coupled with the austerer virtues of the metaphysician and logician. He was a man of action as well as a man of ideas. He was well known for his kindly disposition and his broadmindedness. He possessed not only the sympathy with the poor necessary for a social reformer, but the ability to develop his ideas of social reconstruction on practical lines. Not only did he elaborate a theory of an ideal state, but he also introduced numerous reforms in the actual state of which he was a member. He studied theology very earnestly as a young man and later wrote a great deal on the subject. Even though his liberal views brought him into collision with those on whom his livelihood depended, yet he courageously continued his teaching and writing, always making it his aim to seek for truth. He was a metaphysician of some importance and his treatises on metaphysics are valuable, not only for the original thought which they contain, but also for his important criticisms of Kant. In esthetics his work is by no means without interest, and to the psychology and ethics of his day he made very valuable contributions. But preeminently he was a mathematician and logician. In his

* We regret that owing to limited time and the uncertainties of transatlantic mail service *The Monist* is compelled to go to press without receiving the author's *imprimatur.*

work on mathematical analysis and mathematical logic, he stood out from all the other thinkers of his day. He was a man of many ideas and his intellectual equipment made him able to indicate to his followers the most fruitful lines of inquiry. All through his life he worked for the good of mankind, helping it on in its search for truth.

Bernard Bolzano was born on October 5, 1781, at Prague.[1] He was the fourth son of Bernard Bolzano, an upright and philanthropic member of the Italian community at Prague. His mother was a very pious women. He had a large number of brothers and sisters, the majority of whom perished in childhood; he himself was a sickly child. In his early youth he was very much interested in mathematics and philosophy. His education was of the type usual at the end of the eighteenth century. He tells us that as a child he used to let passion completely overmaster him because he believed that he was raging not at people but at Evil itself. Bolzano was sent to one of the gymnasia of his native city, where he did not distinguish himself very much, and later proceeded to the university there. At the university he studied philosophy and subsequently theology. It was his father's wish that he should be a business man, and though his father finally gave way he showed his disapproval of his son's desire to continue his studies in various ways.

Bolzano had been brought up a Roman Catholic and he was much troubled with doubts as to whether he should take orders. Finally, however, he became convinced that difficult problems, such as the authenticity of the miracles, were not essential parts of the Catholic faith, and as in his opinion the office of priest offered the best opportunity of doing good, he took orders in 1805. At the same time he became doctor of philosophy at Prague University, and

[1] *Lebensbeschreibung des Dr. B. Bolzano mit einigen seiner ungedruckten Aufsätze und dem Bildnisse des Verfassers; eingeleitet und erläutert von dem Herausgeber* (J. M. Fesl), Sulzbach, 1836.

was appointed professor of the philosophic theory of religion.

As professor, Bolzano suffered many cramping indignities which surrounded all teachers in Roman Catholic countries at that time. To a man with Bolzano's sympathies, the position must have been a peculiarly trying one. He had a great love for young people[2] and mixed freely with the students. He was particularly sought after by the students because of his liberal views. His broad-minded interpretation of the dogmas of the Catholic faith, while provoking the distrust of the authorities, recommended him to the younger generation, and he wielded a great influence in their revolutionary schemes and was thought by many to have supported them with an enthusiasm unbecoming in a professor. At any rate, relations between Bolzano and the authorities grew more and more strained, and finally, as he would not recall what they were pleased to call his "heresies," he was dismissed on the grounds that he had "failed grievously in his duties as priest, as preceptor of religion and of youth, and as a good citizen."

After his dismissal from Prague, two ecclesiastical commissions were successively appointed by the Archbishop of Prague to inquire into the orthodoxy of his teaching. In the first commission, the majority declared that Bolzano's teaching was entirely Catholic, but the word "entirely" was deleted at the wish of the minority—which consisted of one person. This decision so enraged the obscurantist party that a large amount of evidence (not a small amount of which was "faked" for the purpose) was collected and put before the second commission. In 1822 Bolzano made two declarations in writing in which he stated that he held it "dangerous, even with the best intentions, for a man to seek and teach new points of view

[2] See A. Wishaupt, *Skizzen aus dem Leben Bolzanos: Beiträge zu seiner Biographie von dessen Arzte,* Leipsic, 1850, pp. 19ff.

as proofs of the truth and divine nature of the Christian Religion."[3] The commission then finally collapsed. Two years later Bolzano was pressed for a public recantation. The Archbishop of Prague brought illicit pressure to bear on him by pleading his affection for him and by declaring that a refusal would bring him to the grave. Bolzano, however, refused to recant publicly, but solemnly declared his orthodoxy in writing.

The main points of his teaching on religion are set out at some length in his *Lehrbuch der Religionswissenschaft*.[4] He defines religion as the aggregate of doctrines which influence man's virtue and happiness. He then proceeds to discuss what seemed to him the most perfect religion, viz., the Catholic faith. His reason for so regarding the Catholic faith is that it is, in his opinion, revealed by God. A religion is divinely revealed, according to Bolzano, if it is morally beneficial and if connected with it there are supernatural events which have no other use than that they serve to demonstrate this religion. In the first chapter the concepts of religion in general, and organized religion in particular, are discussed. In the third chapter he maintains that for a religion to be true it must be revealed, and then he proceeds to enunciate the characteristics of a revelation. In the second volume, he sets out to prove that the Catholic religion possesses the highest moral usefulness and that its origin has the attestation of supernatural occurrences. He discusses the evidence for Christ's miracles and the genuineness of the sources and points out the presence in Christianity of the external characteristic of revelation. He then passes on, in the third volume, to demonstrate in some detail the moral usefulness of the faith. After a discussion of the Catholic doctrine of the sources of knowledge he examines the various doctrines of the

[3] Published 1836 (Sulzbach) with autobiography.
[4] Sulzbach, 1839 (4 volumes).

Catholic church. It is interesting to notice that he regards the doctrine of the Trinity as entirely reasonable, and compares the Father to the All, the Son to humanity, and the Holy Ghost to the individual soul. In the last chapter of this volume Bolzano is concerned with the Catholic system of morals. In his investigation he discusses first Catholic ethics and then the various means of salvation recommended by the church. He examines each of the sacraments in turn.[5]

After his dismissal from Prague, Bolzano wrote a very great deal, but the internal censorship prohibited all publications in his name and even in some cases retained the manuscript. Bolzano once expressed the pious hope that some day he might be allowed to publish some work of a purely mathematical nature! After he left Prague he lived chiefly with friends at Techobuz. He came back, finally, to his native city in 1841 and continued his work with vigor until his death in 1848.

Though it was in mathematics that Bolzano did his most important work, yet in other subjects, notably in political science, his work is of considerable value. He had very great sympathy with the poor and was anxious to abolish class differences. He was convinced that the inadequacy of social organizations was the cause of poverty. He never wrote very much on the matter, but made it the subject of many of his professorial addresses. There is, however, one short manuscript[6] in which he sets out the main points of his political theory. Bolzano himself thought a great deal of this manuscript for he says in the introduction: "And small as is the number of these pages, yet the author thinks he may be allowed to attribute some value to them. Nay, he considers that this little book is

[5] For a complete list of his theological works see Bergmann, *Das philosophische Werk Bernard Bolzanos,* Halle, 1909, p. 214.

[6] "Vom besten Staate, MS. in the Royal Bohemian Museum. For a convenient summary of the MS. see Bergmann, *op. cit.,* pp. 130ff.

the best and most important legacy that he can bequeath to his fellow men if they are willing to accept it."

In Bolzano's ideal state, men and women alike are to have the privilege of voting, but a person is only allowed to vote on a matter of which he has some knowledge and in which he has some interest. Further, the right of voting is liable to forfeiture in the case of misconduct. Any citizen may put forward a suggestion. The suggestion is examined by six independent citizens, each one examining it privately, and it is only rejected if all six of the citizens reject it—and even then it is retained by the state for further reference. If it is not rejected, a general vote is taken, and if there is a majority in favor of it, it goes to a council[7] which is composed of men and women over sixty years of age, who are chosen by the people every three years. The council can only veto the decisions of the people if ninety percent of the council are against it. The government is the administrative body, its members are paid and elected by the people, and there is a strict limit to the length of time that they may remain in office. The government takes special care to prevent private individuals combining in their own interest. Bolzano looked upon war as a dreadful misfortune and in his Utopia war is only to be used as a defensive measure. Bolzano points out that internal revolutions are unlikely, for they arise in general from one of two causes—a bad constitution or poverty. Of these, poverty is to be non-existent and a revolution due to the first cause is improbable because it could only be brought about if the council opposed a change in the constitution which the people considered advisable. But the council in its wisdom would not taunt the people but would give reasons for its decision. It therefore seems unlikely that the people would rise in revolt, all the more because it is early impressed upon the young that a good

[7] The council is called the "*Rat der Geprüften.*"

citizen does not work against the government, for the government's object is to work for the good of the whole state.

One of the most interesting parts of the manuscript deals with the idea of property. In the ideal state property is only desired in so far as the possession of it contributes to the common good. The only valid claim of a man to property is, therefore, that he can make it more useful to the state than any one else could. The fact that a man may possess a certain thing at a certain time is not a necessary or sufficient reason that he shall possess it altogether. The right of inheritance is not recognized. Things such as books, paintings, furniture or jewels, are given to a citizen to use but not to possess. Further, even though he may have established his claim to a certain object, yet, if at any subsequent time another citizen can make more use of it, the title of the first citizen to it is gone. Moreover, the state does not offer any compensation to a man for depriving him of anything. Thus a man whose eyesight has been cured has his glasses taken away and no compensation is made. In all the distribution of goods the government is guided entirely by the principle that the use of a certain thing should be granted to the citizen who can render it most useful to the state as a whole.

The ideals of the state are freedom and equality. There is no unequal distribution of wealth. However there is not an absolute equality of owners, for, as Bolzano points out, the possibility of increasing one's property is a powerful incitement to work. But there are limits beyond which a man cannot increase the extent of his property, and these limits are determined by the consideration of the good of the state as a while. There are "equal" right for all citizens, but the word "equal" is not to be interpreted in any narrow sense. Rather there is an adjustment between the rights of a citizen and his obligations, between his strength

and his need. The government aims at promoting religious freedom. No religion is given preferential treatment by the state. People choose their own ministers of religion and support them. But a new religion may not be preached without permission, for some might not be able to grasp all the consequences of accepting certain doctrines and beliefs. Further, a citizen may change his religion, but he must first bring proof that he has studied with earnestness the principles of the religion he is about to leave, as well as of the one which he desires to embrace.

In the education of children the special aim is the development of the mind. The teachers do not have complete freedom in the choice of what the children are taught. The Council, if it is unanimous, has the power to prevent the teaching of any particular doctrine. The children's books are censored. The censor is responsible directly to the government. And not only the children's books, but all the books in the state are censored strictly.

The question of rewards and punishments in the state is treated in a practical way. Rewards are to consist in public recognition of merit, and punishments are not arranged on a definite plan but are modified so as to suit individual cases. There is however a special proviso that no citizen is under any circumstances to be imprisoned for life.

Bolzano has some very interesting ideas on the occupations of the people in his Utopia. To begin with, the state is to support those who are not fit to work. From those who are fit, the state demands a certain fixed amount of work—the fixed amount, of course, varying from one individual to another. In return for the work the state distributes goods. Citizens are not allowed to waste their time in useless or pernicious occupations—Bolzano considered newspapers pernicious. Neither are they allowed to do things in any but the quickest and most satisfac-

tory way. Thus they are not allowed to thresh with a flail when a threshing machine has been invented, nor, presumably, to walk when there is a tram. One interesting point is that the state is to pay compensation for damage done by nature. Bad weather would quickly lose its terror for farmers in Bolzano's ideal state. Finally, those who wish to devote their lives to art or some branch of learning are supported by the state if they can produce evidence to show that it will be in the state's interest that they shall be employed in this way. The whole theory of the state is peculiarly fresh and in many respects suggestive.

But Bolzano's Utopia is only a practical illustration of his general ethical principles. The guiding principle of his inquiry may be enunciated as follows: Of all possible actions, one should always choose that one which, when all consequences have been considered, produces the greatest amount of good or the least amount of evil, for the human race as a whole, and in this estimate the good of individuals, as such, is to be left out of consideration. But Bolzano points out that if this principle is to be the highest moral law, it would be necessary to frame a definition of *good* and *bad* before any practical applications could be made. Further since he holds that an action is good if it is an action which we ought to perform, he gets back immediately to the question: What ought I to do?[8]

There then remains only the effects of action on the faculty of sensation. Bolzano argues that, since one can excite only either pleasant or unpleasant sensations and since no one would hold that it is one's duty to excite unpleasant sensations, it is obviously one's duty to excite pleasant sensations. By this process of eliminating everything except the faculty of sensation, Bolzano comes to the conclusion that the highest moral duty is the excitement

[8] For an interesting and valuable criticism of Bolzano's assertions and deductions mentioned here, see Bergmann, *op. cit.*, Part V, § 958.

of pleasant sensations. Not the least interesting part of his work in ethics is his criticism of Kant's categorical imperative. He urges the necessity for a modification in Kant's principle and points out the invalidity of Kant's theory that the opposite of a duty involves a contradiction.

Bolzano's work in esthetics is not without interest.[9] His theory of esthetics is the result, not of his own esthetic sensations, but of a painstaking analysis of the abstract idea. His definition of the scope of the subject does not make it coincide with the theory of beauty unless we include in that theory not only the sum total of truths directly concerned with beauty but also all those which stand in such a relation to them that either the former cannot be thoroughly understood without the latter or the latter without the former. To get at his concept of beauty, he eliminates goodness and attractiveness, and by this process obtains a first criterion of beauty, viz., all beauty is pleasant, i. e., it produces pleasure and this pleasure arises solely from the contemplation of the object. Further, since animals are to be excluded from esthetic enjoyment, qualities must be introduced which they do not possess, e. g., intelligence, judgment and reason. Bolzano then comes to the conclusion that it is the growth of these qualities in us that is responsible for the pleasure we find in beauty. Together with the "Ueber den Begriff des Schönen" in the Royal Bohemian Museum, there is another short treatise of Bolzano's in which a theory of laughter is elaborated.[10] Bolzano thought that laughter was caused by the rapid alternation of pleasant and unpleasant sensations and from the fact that animals and infants do not laugh he deduces that laughter is not entirely physical.[11]

In his metaphysics, Bolzano reveals himself as "one of

[9] See *Ueber den Begriff des Schönen,* Prague, 1843.

[10] *Ueber den Begriff des Lächerlichen,* 1818.

[11] See Bergmann, *op. cit.,* Part IV, § 56.

the acutest critics of the Kantian philosophy and the 'idealist' development from Fichte to Hegel."[12] He also did some important original work. His chief book on the subject,[13] entitled *Wissenschaftslehre: Versuch einer ausführlichen und grösstenteils neuen Darstellung der Logik,*[14] is divided into five sections. In the first of these he sets out to prove that objective truth exists and that it is possible for us to have knowledge of it; but he allows that in the development of the science of knowledge, which is the most fundamental of the sciences, it is necessary to use some psychological methods of treatment. In the second part, the "Theory of Elements," he treats successively ideas-in-themselves, their combination into propositions-in-themselves, the theory of true propositions-in-themselves, and finally their combination into syllogisms. He is extremely careful to distinguish between the idea-in-itself and the conceived idea. The concept of a proposition-in-itself is produced by a double abstraction. First the meaning of the proposition and the words conveying the meaning have to be separated from each other, and then one has to forget that the proposition has ever been in anybody's mind. By this means we get to the concept of a proposition-in-itself.

In the distinction that he draws between perception and conception, Bolzano himself says that he owes very much to Kant, but Bolzano disagrees with him in the use he makes of this distinction in his theory of time and space. Bolzano examines in some detail Kant's theory of time and space and his theory of the categories, making some very acute criticisms. After an investigation into the theory of the syllogism and a discussion of the function

[12] A. E. Taylor, *Mind,* October, 1915.

[13] For a criticism of Bolzano's theories see M. Palagyi, *Kant und Bolzano,* Halle, 1905.

[14] Sulzbach, 1837.

of the linguistic expression of a proposition, the "Theory of Elements" closes with a criticism of previous works on the subject. Next Bolzano considers the appearance in the mind of propositions-in-themselves. And it is in this part of his work in particular that we see the extent and depth of his learning. He treats first our subjective ideas, then our judgments, then the relation of our judgments to truth, and finally their certainty and probability. In this investigation Bolzano uses psychological methods to some extent. Then after the fourth part, the "Art of Inventing," he comes at last in the fifth part to the "Science of Knowledge Proper." The book is remarkable as much for its wealth of original thought and the clearness of expression as for the important criticisms of earlier works on the subject.

But important as is Bolzano's work in metaphysics, ethics, esthetics, and theology, it is preeminently as a mathematician that he should be remembered. Now there are two ways of looking at mathematics. One can look upon it as Huxley did: "Mathematics may be compared to a mill of exquisite workmanship, which grinds you stuff to any degree of fineness." On the other hand, one can look upon mathematics as a real and genuine science and then the applications are only interesting in so far as they contain and suggest problems in pure mathematics. From the second point of view the most important business of the mathematician is to examine and strengthen the foundations of mathematics and to purify its methods. In addition to these points of view which may be called the practical and the philosophical, a third point of view has sprung up in the last century which may be called the purely logical point of view. Whitehead describes this new point of view in the words, "Mathematics in its widest significance is the development of all types of formal, necessary, deductive,

reasoning."[15] In this purely logical system, it is proposed to treat any special development of mathematics with the help of a definite, logically connected complex of ideas, and the mathematician is not to be satisfied to solve particular problems with the help of any methods which may casually present themselves, however ingenious these methods may be. Clear definitions and unambiguous axioms must be framed and then by rigorous reasoning the theorems of the subject are to be deduced.

We find examples of the first and second points of view among the Greeks. It is said of Pythagoras that "he changed the occupation with this branch of knowledge into a real science, inasmuch as he contemplated its foundation from a higher point of view and investigated the theorems less materially and more intellectually,"[16] and of Plato that "he filled his writings with mathematical discussions, showing everywhere how much geometry there is in philosophy." Just as mathematics among the Greeks had its origin in the geometry invented by the Egyptians for practical surveying purposes, so the mathematics of the seventeenth and eighteenth century received its stimulus from the practical researches of Kepler, Newton and Laplace. But in this same fragment of Eudemus we find it recorded that Euclid tried to revise the methods used and "put together the elements, arranging much for Eudemus, finishing much for Thaetetus; he moreover subjected to rigorous proofs what had been negligently demonstrated by his predecessors."

This same work that Euclid did for Greek mathematics three hundred years B. C., the new school of nineteenth century mathematicians performed for European mathe-

[15] A. N. Whitehead, *A Treatise on Universal Algebra*, Cambridge, 1898, preface, p. vi.

[16] Extract from a fragment preserved by Proclus; generally attributed to Eudemus of Rhodes who belongs to the peripatetic school and wrote treatises on geometry and astronomy. See extracts in J. T. Merz, *History of European Thought in the Nineteenth Century*, Vol. II, p. 634.

matics. The researches of Newton had suggested a wealth of material for mathematical treatment. Newton and Leibniz had stumbled across the powerful methods of the calculus, which were of tremendous practical importance; but as Klein says, "the naive intuition was especially active during the period of the genesis of the calculus,"[17] and in the great call for powerful methods the theoretical side was almost entirely overshadowed. For example Newton assumed the existence of the velocity of a moving point at every point of its path, not troubling whether, as subsequent investigation has shown to be the case, there might not be continuous functions having no derivative. The great work then of this new school was to investigate the validity of the methods used in the two previous centuries. This was no easy task, and it is only now after one hundred years that the theory of the subject is being put on a logically satisfactory basis. The most important ideas round which the greater part of the work in mathematics centered, are those of continuity and infinity. The importance of these concepts became apparent from the work done on infinite series. A particularly simple example of series, viz., decimal fractions, was in use as early as the sixteenth century, but Leibniz was the first mathematician to have any idea of the importance of series in mathematics. Before his time it had not been realized that an infinite series can only have a meaning under certain circumstances. Unfortunately Leibniz came to the conclusion that the sum of the series

$$1 - 1 + 1 - 1 \ldots \textit{ad inf.}$$

is $\frac{1}{2}$,[18] and so exercised a somewhat baneful influence on

[17] *Evanston Colloquium; Lectures on Mathematics delivered September, 1893*, Lecture VI.

[18] Euler in 1755 (*Instit. Calc. Diff.*) defined the sum of this series to be $\frac{1}{2}$. In the recent theory of divergent series (due in great measure to E. Borel see his *Leçons sur les séries divergentes*, Paris, 1901) one way of defining the formal sum of a divergent series Σa_n is as the limit, when it exists, of $\Sigma a_n x^n$ as x tends to unity through values less than unity. This definition has the

subsequent mathematical developments of the theory of infinite series. However it was left to the genius of Bolzano[19] to enunciate for the first time the necessary and sufficient conditions for the convergence of an infinite series. In 1804 Bolzano published his *Betrachtungen über einige Gegenstände der Elementargeometrie* (Prague), and in 1810 his *Beyträge zu einer begründeteren Darstellung der Mathematik* (Prague). In 1816 he published an important tract on the binomial theorem. In this tract his work on convergency is of great value and his investigation for a real argument (which he everywhere presupposes) is very satisfactory. Bolzano comments on the unrestricted use of infinite series which was common at the time. In 1812 Gauss had published an investigation into the circumstances under which the hypergeometric series converges, and in 1820 Cauchy delivered some extremely important lectures on analysis at the Collège de France, where he was the leader of a group of young mathematicians. Thus Bolzano, Gauss and Cauchy were the pioneers. In his book, *Der binomische Lehrsatz und als Folgerung aus ihm der polynomische und die Reihen, die zur Berechnung der Logarithmen und Exponentialgrössen dienen, genauer als bisher erweisen* (Prague), Bolzano has made a valuable criticism of earlier investigations. It is remarkable that his writings, though of great importance, received comparatively little attention at the time. According to Merz, he had not, like Cauchy, "the art peculiar to the French of refining their ideas and communicating them in

merit of simplicity and also of "consistency," i. e., When the series Σa_n converges, its sum is still the limit as x tends to unity through smaller values, of $\Sigma a_n x^n$ if this limit exists.

Defining the formal sum in this way the sum of the series $1-1+1\ldots$ *ad* inf. is ½.

[19] Accounts of Bolzano's mathematical work were given by Otto Stolz (*Math. Ann.*, Vol. XVIII, 1881, pp. 255-279; Vol. XXII, 1883, pp. 518-519) and on pp. 37-39 of the notes at the end of the reprint of Bolzano's "Rein analytischer Beweis" of 1817 in No. 153 of *Ostwald's Klassiker.*

the most appropriate and taking manner."[20] In his *Rein analytischer Beweis* (1817) Bolzano tells us that it is very much better to publish one's mathematical work in separate treatises; in this way there is more chance of getting acute criticism. Consequently we find his mathematical work scattered about in various small treatises.[21] Also he tells us that one of his treatises had the misfortune not to be noticed by some of the learned periodicals and in others to be criticized only superficially.

In 1842, in the course of some work on the undulatory theory of light, he made a prophecy which is extremely interesting in the light of the invention of spectrum analysis and the researches of Sir W. Huggins, Kirchhoff, and others. He said: "I foresee with confidence that use will hereafter be made of it in order to solve, by observing the changes which the color of stars undergoes in time, the questions as to whether they move, with what velocity they move, how distant they are from us and much else besides." But let us return to the most *important* part of Bolzano's mathematical investigations.

In 1817 Bolzano published a paper we have already mentioned entitled "Rein analytischer Beweis des Lehrsatzes: dass zwischen je zwei Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reele Wurzel der Gleichung liege." This paper is, in a way, his most important work and is a triumph of careful and subtle mathematical analysis. His central theorem, as the title indicates, is as follows: If in an equation $f(x) = 0$, $x = \alpha$ makes $f(x)$ positive and $x = \beta$ makes $f(x)$ negative, then there is at least one real root of the equation $f(x) = 0$ between α and β. Before he begins his constructive work he criticizes very acutely the previous attempts of Lagrange and others. He points out the errors that had

[20] *Op. cit.*, Vol. II, p. 709.

[21] For complete list see Bergmann, *op. cit.*, pp. 213-214.

been made by previous investigators and he emphasizes once more the great importance of freeing mathematical analysis from the intuitional treatment to which it had formerly been subjected. In order to prove his main theorem, Bolzano found it necessary to introduce the concept of the continuity of a function, the notion of the upper limit of a variate and some important work on infinite series. His method is briefly as follows:

1. He introduces the concept of "continuity." A function is said to be "continuous" for the value x if the difference between $f(x+\omega)$ and $f(x)$ can be made less than any assigned number, however small, if only ω is taken sufficiently small.

2. He discusses the convergence of infinite series and makes the following important statement. "If the difference between the value of the sum of the first n terms and the first $n+r$ terms of a series can be made as small as we please, for all values of r, if only we take n large enough, then there is one number X and only one such that the sum of the first p terms approaches ever more and more nearly to X as p increases." Unfortunately his proof of this theorem is not rigorous and his discussion only renders the existence of X probable.

3. From his work on infinite series Bolzano passes on to an extremely important theorem in which he introduces the new idea of an upper limit. And the theorem, as it occurs in this paper, gains in importance from the fact that the method used is one of fundamental importance in analysis. The theorem runs as follows: "If u_n be such a number that the property M holds for all values of x which are less than u_n, and if the property does not hold for all values of x without exception, then of all the numbers u_n satisfying this condition there is one (say U) which is greater than all the others." This theorem, which might appear obvious to those who allow their geometrical in-

tuitions to cloud their mathematical ideas, is proved by Bolzano with great care and completeness. The method used in the proof was used a great deal by Weierstrass and is now known as the "Bolzano-Weierstrass" process. As the method is of such great importance, we will indicate roughly the way it is used in the proof of this theorem. It will be convenient to call x's which have the property M "suitable" x's and x's which do not have the property M "unsuitable" x's; and further to call a number N a "suitable" number if all x's which are less than N have the property M, and to call a number N an "unsuitable" number if there are some values of x, less than N, which do not have the property M. Now it is obvious that there is a positive number D, such that $u_n + D$ is an unsuitable number. Then, bisecting the interval between u_n and $u_n + D$, we get the number $u_n + D/2$; bisecting the interval between u_n and $u_n + D/2$ the number $u_n + D/2^2$; and so on. When either all the numbers $u_n + D/2^r$ for $r = 1, 2, 3 \ldots$ are unsuitable or there is a number R such that $u_n + D/2^R$ is an unsuitable and $u_n + D/2^{R-1}$ a suitable number. In the first case the existence of U is established, U being equal to u_n. In the second case we repeat the process, dividing the interval between $u_n + D/2^{R-1}$ and $u_n + D/2^R$. Again, either all the numbers $u_n + D/2^R + D/2^{R+s}$, $s = 1, 2 \ldots.$ are unsuitable or there is a number S such that $u_n + D/2^R + D/2^{R-S}$ is an unsuitable and $u_n + D/2^R + D/2^{R+S} - 1$ a suitable number. We continue the same process: if it does not terminate we get finally to an infinite series

$$u_n + D/2^R + D/2^S + D/2^T + \ldots$$

and since R, S, T... are positive integers the series obviously satisfies the conditions of the theorem in paragraph (2) above, and so there is a definite limit to which it tends.

this limit being the "upper limit" U in question. The existence-theorem for an upper limit is thus established.

4. Bolzano next attacks the following theorem: "$f(x)$ and $\varphi(x)$ are continuous functions of x and for $x = \alpha$, $f(x) < \varphi(x)$ and for $x = \beta$, $f(x) > \varphi(x)$: then there is a value of x between α and β for which $f(x) = \varphi(x)$." We will indicate the method Bolzano uses to prove it and we shall see exactly why he found it necessary to establish the existence of an "upper limit." Bolzano shows that, since $f(x)$ and $\varphi(x)$ are continuous, there is a number ω such that all numbers less than it satisfy the relation $\varphi(\alpha + \omega) > f(\alpha + \omega)$. Such a number we may call as in paragraph (3) a "suitable" number. Then from a direct application of the theorem about an upper limit he establishes the existence of an upper limit, say U, for all suitable numbers. It is then easy to show that $f(\alpha + U)$ cannot be less than $\varphi(\alpha + U)$ and cannot be greater than $\varphi(\alpha + U)$ and is therefore equal to $\varphi(\alpha + U)$. In this kind of way Bolzano proves the existence of the value of x between α and β giving $f(x) = \varphi(x)$.

5. Finally Bolzano proves that an expression of the form

$$a + bx^m + cx^n + \ldots + px^r,$$

in which $m, n, \ldots r$ are positive integers, is continuous. Then by means of an easy application of a slightly modified form of the theorem in (4) he proves that there is at least one real root between α and β. The whole paper is extremely valuable and it is interesting to see how Bolzano was led from his central theorem to the theorem in (4), to the concept of "continuity" and the idea of an "upper limit," and in the existence-theorem for the upper limit to the question of the convergence of series.

In mathematical logic and in the theory of infinite numbers, Bolzano's work was also of great importance. Bol-

zano's definition of the continuum is of some interest in itself. He defines a continuum as a set of points such that every point has another point also belonging to the set as near to it as we please.[22] This is expressed in modern phraseology by saying that the continuum is a set of points which is "everywhere dense." The name continuum is now used (after Cantor) only for a set of points which is not only "everywhere dense" but also "perfect." A set of points is "perfect" when every convergent sequence has a limit which is itself a number belonging to the set, and conversely when every number is the limit of properly chosen convergent sequences of numbers themselves belonging to the set.[23] Thus Bolzano would call the set of rational numbers a "continuum," but this set is not perfect and is therefore not a "continuum" in the modern sense of the word. In his work on infinite numbers Bolzano anticipated to some extent the work of Georg Cantor. An "infinite" collection is defined to be a collection which has no last term.[24] He proves that the number of natural numbers and the number of real numbers is infinite, and he sees (§ 49) that the number of these two collections is different. Bolzano also recognizes the fact that it is possible to arrange the points in two lines of different lengths so that each point of one collection corresponds to one single point of the other collection and *vice versa,* no point being left without a corresponding point. This brilliant idea of a one-one correspondence went a long way toward dispersing the cloud of mystery which hung over the contemporary infinite number. Leibniz had stated the difficulty quite plainly. Every number can be doubled, he said, therefore the number of natural numbers and the number of even natural numbers is the same. Therefore the whole

[22] *Paradoxien des Unendlichen,* Leipsic, 1851, 2d ed., Berlin, 1889, § 38.

[23] See E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series,* Cambridge, 1907, p. 49.

[24] *Paradoxien des Unendlichen,* § 9.

is equal to the part—which is absurd. Bolzano realized that there is no real contradiction in this. This same idea of the one-one correspondence between points belonging to certain sets of points has led to the modern idea of "reflexiveness" of infinite numbers. The property of "reflexiveness"[25] together with that of "non-inductiveness,"[26] which disposes of all attempts to count up infinite collections or identify the number of terms in an infinite collection with the ordinal number of the last, has removed all serious difficulties and has helped to make it possible to put the concept of an infinite number on a logical foundation.[27] Defining "similar" classes as classes whose terms have a one-one relation to each other and the "cardinal number" or "power" of a class as the class of all similar classes, we see immediately that the class of natural numbers and the class of even natural numbers have the same cardinal numbers. Thus Bolzano was quite right in seeing no contradiction in Leibniz's statements.

From these few references to isolated theorems and statements in Bolzano's work, it is seen that he had most of the ideas essential in the modern view of mathematics, and that in mathematics at least Bolzano's work has been a source of inspiration to those who came after him. Whether in his theology, his ethics, his political science, his metaphysics, or his mathematics, the desire for clearness of concepts was always his aim. Even the parts of his work which are no longer of intrinsic interest, e. g., his esthetics or his theory of laughter, have an interest for us in that they show us the methods he used in seeking

[25] A number is said to be "reflexive" if it is not increased by adding one to it. See B. Russell, *Our Knowledge of the External World as a Field for Scientific Method in Philosophy,* Chicago and London, 1914, p. 190.

[26] A number is said to be "non-inductive" if it does not possess deductive properties. See B. Russell, *op. cit.,* p. 195.

[27] Cf. the definitions "that which cannot be reached by mathematical induction starting from 1" and "that which has parts which have the same number of terms as itself," B. Russell, *The Principles of Mathematics,* Cambridge, 1903, Vol. I, p. 368.

for truth. That there is objective truth and that we can have knowledge of it—this was the thesis which he set before him in his work. In mathematics especially his work was needed, for whereas idealists maintained that mathematics deals only with appearances, empiricists insisted that mathematics could only approximate to the truth. Bolzano's life work was to start mathematicians on the right way to refute both the idealists and the empiricists. His method of strictly logical analysis of the ideas of continuity and the infinite was the clue which was followed up by all the great mathematical logicians and mathematical analysts of the nineteenth century, until finally the fundamental thesis has been proved that all concepts of pure mathematics are wholly logical. Thus Bolzano was one of the first to suspect and in this he was a worthy successor of the great Leibniz. Unlike most mathematicians of his day, Bolzano did not in his thirst for results succumb to D'Alembert's maxim, *Allez en avant, la foi vous viendra.*

We live in days when some of the contradictions and paradoxes which have perplexed the human race since the days of Zeno are being finally cleared up. Do not let us forget the work of Bolzano who with painstaking endeavor sowed the seeds of this great revolution in mathematical ideas.

Dorothy Maud Wrinch.

Cambridge, England.